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International Journal of Solids and Structures 42 (2005) 4393–4413

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

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Unified classification of stability of pin-jointed bar assemblies

H. Deng [a,*], A.S.K. Kwan [b]

[a] *Department of Civil Engineering, Zhejiang University, 38 Zheda Road, Hangzhou 310027, PR China*
[b] *Cardiff School of Engineering, Cardiff University, P.O. Box 925, The Parade, Cardiff CF24 0YF, UK*

Received 1 September 2004; received in revised form 13 January 2005
Available online 17 February 2005

Abstract

Based on the energy criterion and geometrical nonlinearity theory, this paper broadens conventional concepts of structural stability to explain some non-generic stability phenomena of pin-jointed bar assemblies in a unified and coherent way. A novel classification for stability conditions of such kind of structures is put forward, using analysis of the constitution of the tangential stiffness matrix. Some classical issues, including geometrical stability and stability of mechanisms, are re-investigated under this new concept as part of the formal theoretical development. Effects of bars stiffness are introduced into the necessary and sufficient conditions of intrinsic stability (stability of structure devoid of internal forces). The stability conditions for mechanisms, whether they acquire stiffness from self-stressing or external loading, are also probed. The stability of infinitesimal mechanism is expounded through consideration of high-order variations of the potential energy. Some discussions are provided at the end to build up an integrated understanding of stability of pin-jointed bar assemblies.

Keywords: Structural stability; Pin-jointed bar assembly; Static–kinematic analysis; Geometrical stability; Prestressed mechanism; Infinitesimal mechanism

1. Introduction

Structural stability is the capability a system possesses to maintain its current equilibrium state. Although the definition of structural stability is not unique, a simple and intuitively obvious concept can be expressed (Simitses, 1976) as: "As the external causes are applied quasi-statically, the elastic structure deforms and static equilibrium is maintained. If now at any level of the external causes "small" external

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[*] Corresponding author. Tel.: +86 571 8792414; fax: +86 571 8792165.
E-mail address: denghua@zju.edu.cn (H. Deng).

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doi:10.1016/j.ijsolstr.2005.01.009

Nomenclature

$\mathbf{a}_k$	column k of equilibrium matrix $(nj - c \times 1)$
A_k	cross-sectional area of bar k
$\mathbf{A}$	equilibrium matrix $(nj - c \times b)$
$\mathbf{A}'$	updated equilibrium matrix (removing column $\mathbf{a}_k$ out of $\mathbf{A}$) $(nj - c \times b - 1)$
b	number of bars in an assembly
$\mathbf{b}_{\mathrm{L}k}$	linear part of compatibility vector of bar k (1×6)
$\mathbf{b}_{\mathrm{NL}1k}$	nonlinear part of compatibility vector of bar k, including only first-order terms of $\mathbf{d}_k$ (1×6)
$\mathbf{B}$	compatibility matrix $(b \times nj - c)$
$\mathbf{B}_{\mathrm{L}}$	linear part of compatibility matrix $(b \times nj - c)$
$\mathbf{B}_{\mathrm{NL}}$	nonlinear part of compatibility matrix $(b \times nj - c)$
$\mathbf{B}_{\mathrm{NL}1}$	component of $\mathbf{B}_{\mathrm{NL}}$ only including first-order terms of $\mathbf{d}$ $(b \times nj - c)$
c	number of kinematic constraints in an assembly
d_i	ith component of $\mathbf{d}$
$\mathbf{d}$	vector of nodal displacements $(nj - c \times 1)$
$\mathbf{d}_k$	vector of nodal displacements of bar k (6×1)
$\mathbf{e}$	vector of bar elongations $(b \times 1)$
e_k	elongation of bar k
E_k	Young's modulus of bar k
$\mathbf{F}_i$	product–force matrix $(nj - c \times nj - c)$
j	number of joints in an assembly
$\mathbf{K}_0$	linear elastic stiffness matrix $(nj - c \times nj - c)$
$\mathbf{K}_{\mathrm{d}}$	large-displacement stiffness matrix $(nj - c \times nj - c)$
$\mathbf{K}_{\mathrm{g}}$	geometrical stiffness matrix $(nj - c \times nj - c)$
$\mathbf{K}_{\mathrm{T}}$	tangential stiffness matrix $(nj - c \times nj - c)$
L	length of bar
L_k	initial length of bar k
L'_k	elongated length of bar k
m	number of inextensional mechanisms
m_k	linear axial stiffness of bar k
m_k^*	square root of m_k
$\mathbf{M}$	diagonal matrix of bar stiffness $(b \times b)$
$\mathbf{M}^*$	diagonal matrix $(b \times b)$
n	dimensionality of the structure, 2 or 3 respectively for planar or spatial problem
P	concentrated load at node
$\mathbf{p}$	vector of external nodal loads $(nj - c \times 1)$
$\mathbf{p}_{ij}$	product–force vector $(nj - c \times 1)$
q	dimension of generalised coordinates
r	rank of equilibrium matrix
s	number of self-stress states
$\mathbf{S}$	diagonal matrix of singular values $(nj - c \times b)$
$\mathbf{t}$	vector of bar axial forces $(b \times 1)$
$\mathbf{t}_0$	vector of initial bar axial forces $(b \times 1)$
$\mathbf{t}_l$	vector of initial bar axial forces caused by external loads $(b \times 1)$
$\mathbf{u}_i$	ith left singular vector $(nj - c \times 1)$

$\mathbf{U}$	matrix containing a set of left singular vectors $(nj - c \times nj - c)$
$\mathbf{U}_r$	matrix containing modes of extensional deformation $(nj - c \times r)$
$\mathbf{U}_m$	matrix containing modes of inextensional deformation $(nj - c \times m)$
v_{jk}	a component of $\mathbf{V}$
$\mathbf{v}_i$	ith right singular vector $(b \times 1)$
$\mathbf{V}$	matrix containing a set of right singular vectors $(b \times b)$
$\mathbf{V}_r$	matrix containing modes of kinematically compatible extensions $(b \times r)$
$\mathbf{V}_\mathrm{s}$	matrix of self-stress states $(b \times s)$
u_i, v_i, w_i	nodal displacements of joint i
x_i, y_i, z_i	Cartesian coordinates of joint i
α_i, β_j	combination coefficients
$\boldsymbol{\alpha}$, $\boldsymbol{\beta}$	vector of combination coefficients
γ, η	parameters
λ	control parameter
Π	total potential energy of system
θ	rotation of bar at its end joints
diag{}	symbol of diagonal matrix
o()	symbol representing high-order terms of variables
$r()$	symbol representing rank of a matrix

disturbances are applied and the structure reacts by simply performing oscillations about the deformed equilibrium state, the equilibrium is said to be *stable*."

As one important aspect of structural analysis, the foundations of structural stability are usually illustrated in conventional text books by a strut system under a compressive force as shown in Fig. 1(a). It is readily known that such an equilibrium state is conditionally stable, i.e., the vertical strut would buckle when the compressive force increases to a certain value. On the other hand, if the direction of force is upwards, but other conditions remain unchanged, see Fig. 1(b), the system is in an unconditionally stable

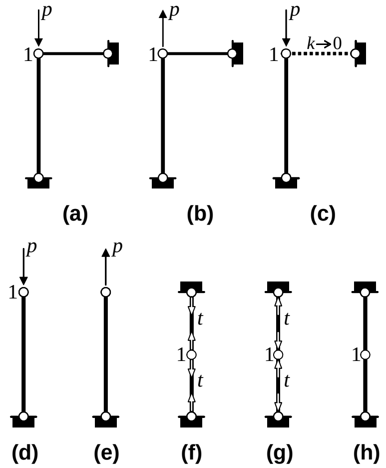

Fig. 1. Different equilibrium states for bar assemblies.

equilibrium state regardless of strength failure. Following this, structural stability depends on the sign (tensile or compressive) and size of the load carried. Clearly, the stability of an equilibrium system is not simply related to loads alone. If, in the limit, the stiffness of horizontal bar of the assembly shown in Fig. 1(a) tends to zero, see Fig. 1(c), then the system becomes unconditionally unstable, so the stability of an equilibrium system must also be affected by member stiffness. Furthermore, although the degeneration of stiffness in the horizontal bar in Fig. 1(c) can be regarded as removal of this bar, as shown in Fig. 1(d), these two cases are classified and treated differently. The system in Fig. 1(d) is known from topology to be an unstable equilibrium because of its variant geometry, and the question of member stiffness never arises.

Although the topic of structural stability could be traced back as early as 1744 to the discussion of a simple pin ended strut by Euler (Bazant and Cedolin, 1991), research on structural stability has overwhelmingly focused on the effects of external load acting on geometrically perfect (i.e., geometrically invariant) system. We hardly find discussion on structural stability of an assembly with geometrical imperfection, because this analysis (also as static–kinematic analysis) is usually treated as a separate topic in structural mechanics. Undoubtedly, the static–kinematic state of an assembly is an important factor affecting the stability as shown in Fig. 1(d). However, we should not thereby conclude that geometrical imperfection must necessarily cause system instability. If the direction of load in the system shown in Fig. 1(d) is reversed, see Fig. 1(e), the system becomes an unconditionally stable equilibrium state according to the above concept of structural stability, even though the assembly is still geometrically variant.

In design, it is important to understand which factors are affecting the stability of the structural system, and what measures to take to avoid instability. However, it may be found from above discussions that there is no absolute clarity on how the interaction of factors such as loads, member stiffness, geometry of assembly, impact on the structural stability of system. We find that loads acting on a geometrically invariant assembly can cause system instability, e.g. Fig. 1(a), but sometimes stability seems independent of load, e.g. Fig. 1(b), and occasionally, loads can even make a geometrically variant assembly stable, e.g. Fig. 1(e).

This paper was primarily motivated by attempt to clarify those unclear phenomena discussed above, and to find a universal system to categorise the relationships among all the interactive factors affecting the structural stability. Our investigation is carried out on the simplest of structural types, i.e. pin-jointed bar assemblies, and by means of the fundamental tool in structural stability, i.e. energy criterion. The layout of the paper is as follows.

Energy criterion of structural stability and properties of stiffness matrix will be briefly reviewed in Section 2. Also, a new analytical expression in vectorial mathematics of the tangential stiffness matrix (especially the geometric stiffness matrix, and thereby allowing consideration of the geometric nonlinearity) of pin-jointed bar assembly is developed in this section.

Section 3 will examine the factors affecting the stability through examining the constitution of the tangential stiffness matrix, and a unified classification for stability conditions of pin-jointed bar assembly is thereby proposed.

In Section 4, the static–kinematic analysis of pin-jointed bar assembly is re-investigated from the viewpoint of structural stability. Necessary and sufficient conditions of "Maxwell's rule" (Maxwell, 1890) as well as the criterion based on rank analysis of equilibrium matrix (Pellegrino and Calladine, 1986) are probed. Supplementary effects of bar stiffness to the intrinsic stability (i.e. structural stability regardless of the effects of internal forces), are also considered. Additionally, a technique for determining "necessary bars" is put forward and strictly proved. Necessary bars are here defined as bars which if removed, or which if they have zero stiffness, will cause the system to be intrinsically unstable.

Stability of mechanisms stiffened by self-stress or external loading is discussed in Section 5. Mechanisms acted by self-stress such as those shown in Fig. 1(f) and (g) have been investigated by eminent scholars over the last two decades or so (Tarnai, 1980; Calladine and Pellegrino, 1991; Kuznetsov, 1991), motivated by configurations with singular geometries such as *tensegrities* (e.g. Calladine, 1978). The term *prestressed mechanism* is also given to these assemblies (Pellegrino, 1990). A determination criterion for these systems was further developed by Calladine and Pellegrino (1991) based on physical explanations. Section 5 provides strict theoretical proofs for Calladine and Pellegrino's criterion, and further irregularities are demonstrated. Mechanisms stiffened by loads, e.g. Fig. 1(e), are also discussed in this section, and a determination criterion for such systems similar to that for prestressed mechanisms is proposed.

In Section 6, we discuss further the assemblies which are infinitesimal mechanisms, e.g. Fig. 1(h), the classification of which is still a matter for debate. From the theory of structural stability and on the basis of discussions of illustrative examples, it is necessary to carry out analysis of high-order variations of potential energy to properly show the characteristics of such kind of assemblies.

Some discussions in the last section of this paper are provided with view to enhance an integrated understanding of the stability of pin-jointed bar assemblies.

2. Energy criterion and tangential stiffness matrix

2.1. Energy criterion

The question of structural stability may be most effectively answered on the basis of the energy criterion. The Lagrange–Dirichlet theorem, which demonstrates that an equilibrium position is *stable* if its potential energy Π is absolute minimal, can be used to ascertain the stability of a conservative system. The potential energy Π is the function of generalised coordinates, such as nodal displacement vector $\mathbf{d}$ and control parameters λ, such as the loading parameter. The increment of potential energy may be expanded into a Taylor Series about the equilibrium state, and hence:

$$\Delta\Pi = \Pi(\mathbf{d} + \delta\mathbf{d}, \lambda) - \Pi(\mathbf{d}, \lambda) = \delta\Pi + \delta^2\Pi + \delta^3\Pi + \delta^4\Pi + \cdots, \tag{1}$$

in which $\delta\Pi$, $\delta^2\Pi$,... are respectively the first, second, ... variations of the potential energy. They can be expressed as

$$\delta\Pi = \frac{1}{1!}\sum_{i=1}^{q} \frac{\partial\Pi(d_1, \ldots, d_q, \lambda)}{\partial d_i}\delta d_i, \tag{2a}$$

$$\delta^2\Pi = \frac{1}{2!}\sum_{i=1}^{q}\sum_{j=1}^{q} \frac{\partial^2\Pi(d_1, \ldots, d_q, \lambda)}{\partial d_i \partial d_j}\delta d_i \delta d_j, \tag{2b}$$

$$\delta^3\Pi = \frac{1}{3!}\sum_{i=1}^{q}\sum_{j=1}^{q}\sum_{k=1}^{q} \frac{\partial^3\Pi(d_1, \ldots, d_q, \lambda)}{\partial d_i \partial d_j \partial d_k}\delta d_i \delta d_j \delta d_k \ldots, \tag{2c}$$

where d_i is the ith component of $\mathbf{d}$, and q is the dimension of generalised coordinates. The conditions of equilibrium are

$$\delta\Pi = 0 \quad \text{for all } \delta\mathbf{d}. \tag{3}$$

Normally, the second-order variation term $\delta^2\Pi$ is adequate to ascertain the state of stability. If the expression in Eq. (2b) is given in terms of matrix notation, the equilibrium state of an assembly is

$$\text{stable} \quad \text{if } \delta^2\Pi = \frac{1}{2}\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_{\mathrm{T}}\,\delta\mathbf{d} > 0 \quad \text{for all vectors } \delta\mathbf{d}; \tag{4}$$

$$\text{critical} \quad \text{if } \delta^2\Pi = \frac{1}{2}\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_{\mathrm{T}}\,\delta\mathbf{d} = 0 \quad \text{for one or more vectors } \delta\mathbf{d};\ \text{or} \tag{5}$$

$$\text{unstable} \quad \text{if } \delta^2\Pi = \frac{1}{2}\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_{\mathrm{T}}\,\delta\mathbf{d} < 0 \quad \text{for one or more vectors } \delta\mathbf{d}; \tag{6}$$

where $\mathbf{K}_{\mathrm{T}}$ is the tangential stiffness matrix of system at the current equilibrium state (also called the Hessian matrix of the total potential energy function (Thompson and Hunt, 1973)). $\mathbf{K}_{\mathrm{T}}$ also represents the incremental mechanical properties of system characterised by the relationship

$$\mathbf{K}_{\mathrm{T}}\,\delta\mathbf{d} = \delta\mathbf{p}, \tag{7}$$

where $\mathbf{p}$ is the vector of nodal loads.

It can be seen from Eqs. (4)–(6) that the expression for $\delta^2\Pi$ is a quadratic form of $\mathbf{K}_{\mathrm{T}}$. From the theory of linear algebra, the stability of a system in a certain equilibrium state can be further reflected by the properties of tangential stiffness matrix $\mathbf{K}_{\mathrm{T}}$, and in particular, whether $\mathbf{K}_{\mathrm{T}}$ is of full rank.

It should be noted that the critical case in Eq. (5) requires deeper investigation. In this case, the stability of system would depend on the nature of higher-order variations of Π for those vectors $\delta\mathbf{d}$ which led to $\delta^2\Pi = 0$. When $\delta^2\Pi = 0$ for one or more $\delta\mathbf{d}$, the system will actually be stable if $\delta^3\Pi = 0$ and $\delta^4\Pi > 0$ for all those $\delta\mathbf{d}$. In special cases, even higher-order variations will have to be examined.

2.2. Basic equations for pin-jointed bar assembly

Throughout the paper, we shall deal with elastic assembly with j joints connected by b pin-jointed bars. A total number of c kinematic constraints prevent some joints from moving in certain directions. Hence, the maximum degree of freedom of the assembly is $nj - c$, where n is 2 or 3 respectively for planar or spatial problem. Three basic relationships need to be satisfied to guarantee an equilibrium state for an assembly.

By means of virtual works principle, the *equilibrium* relationship can be expressed as

$$\delta\mathbf{e}^{\mathrm{T}}\mathbf{t} - \delta\mathbf{d}^{\mathrm{T}}\mathbf{p} = \mathbf{0}, \tag{8}$$

where $\mathbf{t}$ is the vector of real bar axial forces, $\delta\mathbf{e}$ is the vector of virtual bar elongations, and $\delta\mathbf{d}$ is vector of virtual nodal displacements.

If we regard the bars to be made of material with linear elastic stress–strain relationship, then the *constitutive* relationship can be expressed in total, and incremental, forms as

$$\mathbf{t} = \mathbf{t}_0 + \mathbf{M}\mathbf{e}, \tag{9}$$

$$\delta\mathbf{t} = \mathbf{M}\,\delta\mathbf{e}, \tag{10}$$

where $\mathbf{M} = \mathrm{diag}\{m_1,\ldots,m_k,\ldots,m_b\}$ is a diagonal matrix and $m_k = E_kA_k/L_k$. E_k, A_k, and L_k are respectively the Young's modulus, cross-sectional area and unstrained length of bar k, while $\mathbf{t}_0$ is the vector of initial bar axial forces.

The kinematic vectors $\mathbf{d}$ and $\mathbf{e}$ have to satisfy the *compatibility* equation

$$\mathbf{B}\,\delta\mathbf{d} = \delta\mathbf{e}, \tag{11}$$

where $\mathbf{B}$ is the compatibility matrix of assembly. Eq. (11) is expressed in incremental term to allow for geometric nonlinearity in the analysis of structural stability.

2.3. Geometrically nonlinear compatibility matrix

Consider a bar k as shown in Fig. 2, where the coordinates of its two end joints, i and j, at initial equilibrium state are (x_i, y_i, z_i) and (x_j, y_j, z_j) respectively. Let the vector of nodal displacements (coordinate increments) in the deformed equilibrium state be

$$\mathbf{d}_k = \{u_i, v_i, w_i, u_j, v_j, w_j\}^{\mathrm{T}}. \tag{12}$$

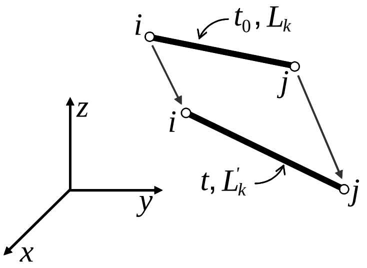

Fig. 2. A bar element.

The lengths of bar k respectively at initial, and deformed, equilibrium states are given by

$$L_k = \{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2\}^{1/2}, \tag{13}$$

$$L_k' = \{(x_i + u_i - x_j - u_j)^2 + (y_i + v_i - y_j - v_j)^2 + (z_i + w_i - z_j - w_j)^2\}^{1/2}. \tag{14}$$

Applying series expansion, the elongation of bar k can be obtained by

$$e_k = L_k' - L_k = \gamma + \eta/2 + \mathrm{o}(\mathbf{d}_k^3), \tag{15a}$$

where

$$\gamma = \{(x_i - x_j)(u_i - u_j) + (y_i - y_j)(v_i - v_j) + (z_i - z_j)(w_i - w_j)\}/L_k, \tag{15b}$$

$$\eta = \{(u_i - u_j)(u_i - u_j) + (v_i - v_j)(v_i - v_j) + (w_i - w_j)(w_i - w_j)\}/L_k, \tag{15c}$$

and $\mathrm{o}(\mathbf{d}_k^3)$ represents the third- and higher-order terms of nodal displacements. Eq. (15a) can be expressed in vector notation as

$$e_k = (\mathbf{b}_{\mathrm{L}k} + \tfrac{1}{2}\mathbf{b}_{\mathrm{NL1}k} + \mathrm{o}(\mathbf{d}_k^2))\mathbf{d}_k, \tag{16}$$

where

$$\mathbf{b}_{\mathrm{L}k} = (1/L_k)\{(x_i - x_j), (y_i - y_j), (z_i - z_j), -(x_i - x_j), -(y_i - y_j), -(z_i - z_j)\}, \tag{17}$$

$$\mathbf{b}_{\mathrm{NL1}k} = (1/L_k)\{(u_i - u_j), (v_i - v_j), (w_i - w_j), -(u_i - u_j), -(v_i - v_j), -(w_i - w_j)\}. \tag{18}$$

Differentiating both sides of Eq. (16) leads to

$$\mathrm{d}e_k = (\mathbf{b}_{\mathrm{L}k} + \mathbf{b}_{\mathrm{NL1}k} + \mathrm{o}(\mathbf{d}_k^2))\mathrm{d}\mathbf{d}_k. \tag{19}$$

With similar relationships for all the bars in assembly, the compatibility relationship of Eq. (11) becomes

$$\mathbf{B}\,\delta\mathbf{d} = (\mathbf{B}_{\mathrm{L}} + \mathbf{B}_{\mathrm{NL}})\,\delta\mathbf{d} = (\mathbf{B}_{\mathrm{L}} + \mathbf{B}_{\mathrm{NL1}} + \mathrm{o}(\mathbf{d}^2))\,\delta\mathbf{d} = \delta\mathbf{e}, \tag{20}$$

where $\mathbf{B}_{\mathrm{L}}$ and $\mathbf{B}_{\mathrm{NL}}$ are the linear and nonlinear parts of the compatibility matrix respectively, and $\mathbf{B}_{\mathrm{NL1}}$ is the component of $\mathbf{B}_{\mathrm{NL}}$ including only first-order terms of $\mathbf{d}$. $\mathbf{B}_{\mathrm{L}}$ is usually named as "the constraint Jacobian matrix" in conventional static–kinematic analysis (e.g. Kuznetsov, 1991). The exact forms of $\mathbf{B}_{\mathrm{L}}$ and $\mathbf{B}_{\mathrm{NL1}}$ are given below:

$$\mathbf{B}_{\mathrm{L}} = \begin{array}{cc} \begin{array}{ccccccccc} & & \text{joint } i & & & & \text{joint } j & & \end{array} & \\ \left\{\begin{array}{ccccccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & (x_i - x_j)/L_k & (y_i - y_j)/L_k & (z_i - z_j)/L_k & \cdots & -(x_i - x_j)/L_k & -(y_i - y_j)/L_k & -(z_i - z_j)/L_k & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right\}_{\substack{b\times \\ (nj-c)}} & \begin{array}{c} \\ \\ \text{bar } k \\ \\ \\ \end{array} \end{array} \tag{21}$$

$$\begin{array}{ccc}
 & \begin{array}{ccccccccc} & & \text{joint } i & & & & \text{joint } j & & \end{array} & \\
\mathbf{B}_{\mathrm{NL1}}= & \left\{\begin{array}{ccccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & (u_i-u_j)/L_k & (v_i-v_j)/L_k & (w_i-w_j)/L_k & \cdots & -(u_i-u_j)/L_k & -(v_i-v_j)/L_k & -(w_i-w_j)/L_k & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right\}_{b\times(nj-c)} & \text{bar } k
\end{array} \tag{22}$$

It should be noted that $\mathbf{B}_{\mathrm{NL1}}$ is a linear function of $\mathbf{d}$, i.e. $\mathbf{B}_{\mathrm{NL1}}(\mathbf{d})$.

2.4. Analytical expression of tangential stiffness matrix

When $\delta\mathbf{e}$ in Eq. (8) is substituted by $\mathbf{B}\,\delta\mathbf{d}$ (Eq. (11)), and the common factor of $\delta\mathbf{d}^{\mathrm{T}}$ (arbitrary and non-null) is removed, we obtain the equilibrium equation of the system

$$\mathbf{B}^{\mathrm{T}}\mathbf{t}-\mathbf{p}=\mathbf{0}, \tag{23}$$

and this equilibrium equation is established in the deformed configuration and not the initial configuration, as opposed to in geometrically linear analysis where $\mathbf{B}$ consists only of $\mathbf{B}_{\mathrm{L}}$.

We can obtain the variation expression of equilibrium equation from Eq. (23) as

$$\delta\mathbf{B}^{\mathrm{T}}\mathbf{t}+\mathbf{B}^{\mathrm{T}}\,\delta\mathbf{t}=\delta\mathbf{p}, \tag{24}$$

and with Eqs. (10), (11) and (20), Eq. (24) can be expressed as

$$\delta(\mathbf{B}_{\mathrm{L}}+\mathbf{B}_{\mathrm{NL}})^{\mathrm{T}}\mathbf{t}+(\mathbf{B}_{\mathrm{L}}+\mathbf{B}_{\mathrm{NL}})^{\mathrm{T}}\mathbf{M}(\mathbf{B}_{\mathrm{L}}+\mathbf{B}_{\mathrm{NL}})\,\delta\mathbf{d}=\delta\mathbf{p}. \tag{25}$$

Since $\mathbf{B}_{\mathrm{L}}$ is dependent only on the initial assembly geometry, and independent of $\mathbf{d}$, then $\delta\mathbf{B}_{\mathrm{L}}=\mathbf{0}$. Re-arrangement of Eq. (25) into a brief form and comparing with Eq. (7), we can obtain the analytical expression of the tangential stiffness matrix of the overall assembly as

$$\mathbf{K}_{\mathrm{T}}=\mathbf{K}_0+\mathbf{K}_{\mathrm{g}}+\mathbf{K}_{\mathrm{d}}, \tag{26}$$

where

$$\mathbf{K}_0=\mathbf{B}_{\mathrm{L}}^{\mathrm{T}}\mathbf{M}\mathbf{B}_{\mathrm{L}}, \tag{27}$$

$$\mathbf{K}_{\mathrm{g}}\,\delta\mathbf{d}=\delta\mathbf{B}_{\mathrm{NL}}^{\mathrm{T}}\mathbf{t}, \tag{28}$$

$$\mathbf{K}_{d}=\mathbf{B}_{\mathrm{L}}^{\mathrm{T}}\mathbf{M}\mathbf{B}_{\mathrm{NL}}+\mathbf{B}_{\mathrm{NL}}^{\mathrm{T}}\mathbf{M}\mathbf{B}_{\mathrm{L}}+\mathbf{B}_{\mathrm{NL}}^{\mathrm{T}}\mathbf{M}\mathbf{B}_{\mathrm{NL}}. \tag{29}$$

Consider the tangential stiffness matrix at the initial equilibrium state where $\mathbf{d}=\mathbf{0}$ and $\mathbf{t}=\mathbf{t}_0$. Since $\mathbf{d}=\mathbf{0}$ in this state, it can be seen from Eqs. (18) and (22) that $\mathbf{B}_{\mathrm{NL}}=\mathbf{0}$, and from Eq. (29), $\mathbf{K}_{\mathrm{d}}$ is thus also equal to a null matrix. As for $\mathbf{K}_{\mathrm{g}}$, only the first-order terms in $\mathbf{B}_{\mathrm{NL}}$, i.e. $\mathbf{B}_{\mathrm{NL1}}$ will remain, and the high-order terms (which relate to $\mathbf{d}$) are equal to zero. Hence, at the initial equilibrium state of $\mathbf{t}=\mathbf{t}_0$, the tangential stiffness matrix can expressed as

$$\mathbf{K}_{\mathrm{T}}=\mathbf{K}_0+\mathbf{K}_{\mathrm{g}}, \tag{30}$$

where

$$\mathbf{K}_{\mathrm{g}}\,\delta\mathbf{d}=\delta\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}\mathbf{t}_0=\delta\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{d})\mathbf{t}_0. \tag{31}$$

The above Eq. (30) is actually the tangential stiffness matrix in updated Lagrangian (UL) formulation in the theory of nonlinear finite element method. The two parts, $\mathbf{K}_0$ and $\mathbf{K}_{\mathrm{g}}$, are usually named as the “linear elastic stiffness matrix” and “geometrical stiffness matrix” respectively.

2.5. Static and kinematic properties of $\mathbf{B}_L$

The linear part of compatibility matrix, $\mathbf{B}_\mathrm{L}$, is actually the transpose of the equilibrium matrix **A** in the initial equilibrium state (e.g. Pellegrino and Calladine, 1986). **A** imparts static and kinematical information of assembly in its initial equilibrium configuration. Two important parameters can be determined from its rank r: $s = b - r$ is the number of self-stress states; and $m = nj - c - r$ is the number of inextensional mechanisms.

The value of r can be determined in a number of ways, but the use of "singular value decomposition (SVD)" (Pellegrino, 1993) on the equilibrium matrix would also give orthogonal sets of m inextensional mechanisms and s states of self-stress, as follows

$$\mathbf{A} = \mathbf{USV}^\mathrm{T}, \tag{32}$$

where $\mathbf{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{nj-c}\}$ consists of a set of left singular vectors, $\mathbf{V} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_b\}$ contains a set of right singular vectors, and a set of singular values is found in the first r non-zero diagonal elements of **S**.

The singular vectors, all with unit norm, can be grouped into the following sub-matrices

$$\begin{aligned} &\mathbf{U}_r = \{\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_r\}; \quad \mathbf{U}_m = \{\mathbf{u}_{r+1}, \ldots \mathbf{u}_{nj-c}\} \\ &\mathbf{V}_r = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_r\}; \quad \mathbf{V}_s = \{\mathbf{v}_{r+1}, \ldots \mathbf{v}_b\} \end{aligned} \tag{33}$$

which have the following interpretations (e.g. Kumar and Pellegrino, 2000):

$\mathbf{U}_r$	contains modes of extensional deformation (i.e. loads that can be equilibrated by the structure in its current configuration);
$\mathbf{U}_m$	contains modes of inextensional deformation, i.e. mechanisms (i.e. loads that cannot be equilibrated);
$\mathbf{V}_\mathrm{r}$	contains sets of kinematically compatible extensions corresponding, through the singular values, to the extensional modes in $\mathbf{U}_\mathrm{r}$ (i.e. bar forces in equilibrium with the external loads in $\mathbf{U}_\mathrm{r}$);
$\mathbf{V}_\mathrm{s}$	contains sets of kinematically incompatible extensions (i.e. states of self-stress).

The subspaces span by **U** and **V** have dual statical and kinematical interpretations because the equilibrium and compatibility matrices are transposes of each other.

3. Classification for stability conditions of pin-jointed bar assembly

From energy criterion described above, structural stability of a system can be further ascertained from the properties of its tangential stiffness matrix at the equilibrium state under investigation. Hence, factors affecting the stability of a pin-jointed bar assembly can be analysed from the constitution of the tangential stiffness matrix.

From Eq. (27), the linear elastic tangential stiffness matrix $\mathbf{K}_0$ relates to two parameters, $\mathbf{B}_\mathrm{L}$ and **M**. $\mathbf{B}_\mathrm{L}$ is defined by the geometry (location of joints) and topology (linkage of bars) of the assembly, while **M** is related with the axial stiffness of bars. On the other hand, the geometrical stiffness matrix $\mathbf{K}_\mathrm{g}$ indicates the effect of initial internal forces on the nodal displacements, and from Eq. (31), $\mathbf{K}_\mathrm{g}$ has direct relationships with the internal forces $\mathbf{t}_0$ and the first-order nonlinear compatibility matrix $\mathbf{B}_{\mathrm{NL1}}$ which is itself dependent on the geometry of assembly.

The factors which affect the stability of a pin-jointed bar assembly can be concluded as three basic aspects: geometry and topology of assembly; stiffness of bars; and internal bar forces. It should be noted that the first two aspects are intrinsic to an assembly, but internal forces are not unique for a system because they are changeable with external loads or self-stress. The stability conditions of pin-jointed bar assemblies can now be classified in terms of the constitution of the tangential stiffness matrix in five separate cases as follows.

Case 1: if $\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_0\delta\mathbf{d} > 0$ and $\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_g\delta\mathbf{d} \geqslant 0$ (with internal forces) for all $\delta\mathbf{d}$, then the system is stable.
Case 2: if $\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_0\delta\mathbf{d} > 0$ and $\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_g\delta\mathbf{d} = 0$ (without internal forces) for all $\delta\mathbf{d}$, then the system is stable.
Case 3: if $\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_0\delta\mathbf{d} > 0$ for all $\delta\mathbf{d}$, and $\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_{\mathrm{g}}\delta\mathbf{d} < 0$ for some $\delta\mathbf{d}$, but $\delta\mathbf{d}^{\mathrm{T}}(\mathbf{K}_0+\mathbf{K}_{\mathrm{g}})\delta\mathbf{d} > 0$ for those $\delta\mathbf{d}$ which caused $\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_{\mathrm{g}}\delta\mathbf{d} < 0$, then the system is stable.
Case 4: if $\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_0\delta\mathbf{d} = 0$ for some $\delta\mathbf{d}$ and $\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_{\mathrm{g}}\delta\mathbf{d} > 0$ for those same $\delta\mathbf{d}$, but for all remaining $\delta\mathbf{d}$, $\delta\mathbf{d}^{\mathrm{T}}(\mathbf{K}_0+\mathbf{K}_{\mathrm{g}})\delta\mathbf{d} > 0$, then the system is stable.
Case 5: if $\delta\mathbf{d}^{\mathrm{T}}(\mathbf{K}_0+\mathbf{K}_{\mathrm{g}})\delta\mathbf{d} = 0$ for some $\delta\mathbf{d}$, and $\delta\mathbf{d}^{\mathrm{T}}(\mathbf{K}_0+\mathbf{K}_{\mathrm{g}})\delta\mathbf{d} > 0$ for all remaining $\delta\mathbf{d}$, the system is in critical stability. However, if higher-order variations of potential energy for all those $\delta\mathbf{d}$ which led to $\delta\mathbf{d}^{\mathrm{T}}(\mathbf{K}_0+\mathbf{K}_{\mathrm{g}})\delta\mathbf{d} = 0$ are greater than zero, then the system is still stable.

The linear elastic stiffness matrix $\mathbf{K}_0$ could not be negative definite (Fung, 1965), i.e. the quadratic form $\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_0\delta\mathbf{d}$ cannot be less than zero. In conventional understanding, singularity of $\mathbf{K}_0$ (i.e. $\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_0\delta\mathbf{d} = 0$ for some $\delta\mathbf{d}$) arises from two situations: the deficiency of bar stiffness (e.g. Fig. 1(c)) or the imperfection of geometry (e.g. Fig. 1(d) or (e)). For an assembly without internal forces (i.e. no contribution from the geometric stiffness matrix $\mathbf{K}_{\mathrm{g}}$), stability analysis is confined to discussion on $\mathbf{K}_0$, and this topic is defined as a problem of "intrinsic stability" in this paper.

Compared to $\mathbf{K}_0$, the geometric stiffness matrix $\mathbf{K}_{\mathrm{g}}$ is much more "active". $\mathbf{K}_{\mathrm{g}}$ may be positive or negative definite depending on the properties of internal forces. In conventional view, if compressive force is predominant in an assembly, $\mathbf{K}_{\mathrm{g}}$ may become negative definite. On the other hand, $\mathbf{K}_{\mathrm{g}}$ may be positive definite for an assembly in a mainly tensile state. This can be illustrated in Table 1 where we examine the properties of the tangential stiffness matrices of the assemblies shown in Fig. 1 and roughly determine their stability according to the five stability conditions above.

Table 1
Stability of equilibratory assemblies shown in Fig. 1

Assembly (see Fig. 1)	$\delta\mathbf{d}$	$\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_0\delta\mathbf{d}$	$\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_{\mathrm{g}}\delta\mathbf{d}$	Result (criteria yielded)
(a)	x_1	>0	<0	Conditionally stable (Case 3)
	y_1	>0	<0	
(b)	x_1	>0	>0	Stable (Case 1)
	y_1	>0	>0	
(c)	x_1	=0	<0	Unstable (Case 4)
	y_1	>0	<0	
(d)	x_1	=0	<0	Unstable (Case 4)
	y_1	>0	<0	
(e)	x_1	=0	>0	Stable (Case 4)
	y_1	>0	>0	
(f)	x_1	=0	>0	Stable (Case 4)
	y_1	>0	>0	
(g)	x_1	=0	<0	Unstable (Case 4)
	y_1	>0	<0	
(h)	x_1	=0	=0	Stable (Case 5)
	y_1	>0	=0	

x_1 and y_1 denote respectively the displacements of Joint 1 in horizontal and vertical directions.

In actual fact, Cases 1 and 3 are the traditional topics of structural stability based on a geometrically perfect assembly. If the effects of bar stiffness are disregarded, Cases 2 is purely an issue of so-called "static–kinematic analysis." Cases 4 and 5 concern the structural stability of "mechanism" (kinematically indeterminate assembly). Table 1 shows that the factors which affect the stability of an assembly are actually interactional. Emphasis of this paper is on the last two groups and we aim to find general rules to explain the interactions between those parameters.

4. Intrinsic stability

Analysis of *intrinsic stability* determines the nature of stability of an assembly devoid of internal forces. This is not a common topic in the discussion of structural stability, but actually it is conventionally investigated as a "static–kinematic analysis" if the effects of bar stiffness are disregarded. Although "static–kinematic analysis" is also called "geometrical stability" (e.g. Kuznetsov, 1999, 2000), the majority of such studies have begun with purely geometrical views, and not from the viewpoint of structural stability. Research on this issue can be traced back to the work of Maxwell (1890), who defined the static and kinematic behaviour of pin-jointed bars system purely by a simple relationship between number of joints, bars and kinematic constraints. Much later, a more general analysis (Pellegrino and Calladine, 1986) based on decomposition of the equilibrium matrix emerged, as discussed in Section 2, which takes into account the geometry and topology of the assembly.

According to the stability condition of case 2 in Section 3, an intrinsically stable system should have

$$\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_0\,\delta\mathbf{d} > 0 \quad \text{for all } \delta\mathbf{d}, \tag{34}$$

which is equivalent to saying that matrix $\mathbf{K}_0$ is positive definite or the rank of $\mathbf{K}_0$ is full, i.e., $r(\mathbf{K}_0) = nj - c$. Since the bar stiffness is also related to $\mathbf{K}_0$, it must be concerned in the investigation of intrinsic stability of an assembly. Hence, Pellegrino and Calladine's approach is insufficient. The linear stiffness matrix $\mathbf{K}_0$ in Eq. (27) can be further factorised as:

$$\mathbf{K}_0 = \mathbf{B}_{\mathrm{L}}^{\mathrm{T}}\mathbf{M}\mathbf{B}_{\mathrm{L}} = \mathbf{A}\mathbf{M}\mathbf{A}^{\mathrm{T}} = \mathbf{A}(\mathbf{M}^{*}\mathbf{M}^{*\mathrm{T}})\mathbf{A}^{\mathrm{T}} = (\mathbf{A}\mathbf{M}^{*})(\mathbf{A}\mathbf{M}^{*})^{\mathrm{T}}, \tag{35}$$

where $\mathbf{M}^*$ is also a diagonal matrix whose diagonal elements $m_k^* = \sqrt{m_k}$ for $k = 1,2,\ldots,b$.

Since the diagonal elements of matrix $\mathbf{M}$ represent the stiffness of the individual bars, they must all be greater than zero. Since $\mathbf{M}^*$ is a diagonal matrix, the product $\mathbf{A}\cdot\mathbf{M}^*$ is obtained merely by multiplying each diagonal element m_k^* with its corresponding column in $\mathbf{A}$, and hence the rank of $\mathbf{A}\cdot\mathbf{M}^*$ takes its value from the rank of $\mathbf{A}$. Furthermore, since the rank of a matrix $\mathbf{X}$ is same as that of $\mathbf{X}\cdot\mathbf{X}^{\mathrm{T}}$ (Jennings, 1977), then

$$r(\mathbf{K}_0) = r[(\mathbf{A}\mathbf{M}^{*})(\mathbf{A}\mathbf{M}^{*})^{\mathrm{T}}] = r(\mathbf{A}\mathbf{M}^{*}) = r(\mathbf{A}) = r. \tag{36}$$

Hence, whether $\mathbf{K}_0$ is positive definite can be judged from the rank of equilibrium matrix $\mathbf{A}_{nj-c\times b}$. Three possibilities follow from this.

(i) For $b < nj - c$, and hence clearly

$$r(\mathbf{K}_0) = r(\mathbf{A}) = r \leqslant b < nj - c. \tag{37}$$

The assembly has insufficient bars and is thus unstable with a deficient rank of $\mathbf{K}_0$. Eq. (37) in fact reflects Maxwell's rule (Maxwell, 1890).

(ii) For $b \geqslant nj - c$ but $r < nj - c$, then

$$r(\mathbf{K}_0) = r(\mathbf{A}) = r < nj - c, \tag{38}$$

and the system is also unstable because some of the bars are actually "ineffective" in providing stability.

(iii) For $b \geqslant nj - c$ and $r = nj - c$ then

$$r(\mathbf{K}_0) = r(\mathbf{A}) = r = nj - c, \tag{39}$$

and the matrix $\mathbf{K}_0$ is positive definite, so the system is stable.

In the above analysis of pin-jointed bar assembly, the criterion of geometrical stability (since it is irrespective of the bar stiffness) is simply whether $r = nj - c$. However, it should be noted that the analysis is based on the presumption that the diagonal matrix $\mathbf{M}$ of member stiffness is of full rank, which is undoubtedly regarded to be the case in conventional view. However, in practical numerical structural analysis, singularity of $\mathbf{M}$ may occur in some conditions, such as when the linear stiffness of a bar tends significantly to zero in magnitude, or even zero because no value has been assigned. Technically, the removal of a bar (Fig. 1(d)) and setting the axial stiffness of that bar to zero (Fig. 1(c)) belong to different categories. The former is about geometry, and the latter is physical. Furthermore, the associated mathematical treatments are also distinct. The two processes, however, produce the same effects on the intrinsic stability of the assembly. A mathematical explanation demonstrates this.

The equilibrium matrix $\mathbf{A}$ $(= \mathbf{B}_{\mathrm{L}}^{\mathrm{T}})$ of a pin-jointed bar assembly can be written as

$$\mathbf{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{k-1}, \mathbf{a}_k, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_b\}, \tag{40}$$

where vector $\mathbf{a}_k$ is kth column of $\mathbf{A}$ and represents the equilibrium contribution from bar k. If the bar k is removed, the equilibrium matrix of the new reduced assembly becomes

$$\mathbf{A}' = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{k-1}, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_b\}. \tag{41}$$

On the other hand, if the stiffness of bar k is set to zero, i.e. $m_k = m_k^* = 0$, but other bars continue to have non-zero stiffness, then the product $\mathbf{A} \cdot \mathbf{M}^*$ contains a null column related to bar k, and therefore $\mathbf{A} \cdot \mathbf{M}^*$ has the same rank as matrix $\mathbf{A}'$,

$$r(\mathbf{A}\mathbf{M}^*) = r(\mathbf{A}'). \tag{42}$$

There is therefore no difference in the stability of the assembly resulting from either bar removal or setting the stiffness of the bar to zero.

The conventional view is that the removal of one or two bars in a highly redundant assembly is unlikely to cause a mechanism. It is also known that while statical redundancy is a good assurance of geometric stability, it is not a guarantee. Furthermore, even in highly redundant and geometrically stable structures, the injudicious removal of one bar can lead to a collapse. For a kinematically determinate but statically indeterminate system, we know the maximum number of bars that *could* be removed, without triggering a mechanism, is s. For example, Fig. 3 shows a planar truss with 10 bars, and $s = 2$. While two bars could be removed without a kinematic change, we cannot choose them arbitrarily. Some bars, e.g. bars 7, 8 and 10, are crucial for maintaining geometric stability, and we thus define these as "necessary bars" for the

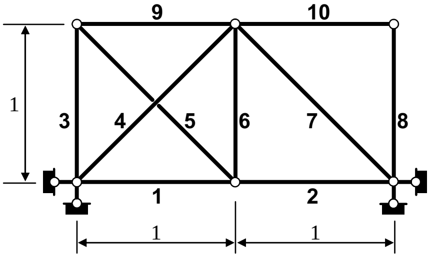

Fig. 3. Ten-bar truss.

assembly. The identity of the necessary bars for a simple structure like that shown in Fig. 3 is not difficult, but there remains the need of a systematic approach, especially for a complex assembly.

Obviously, one straight forward method for identifying necessary bar is to remove each bar in turn, and calculate the rank of the new equilibrium matrix $\mathbf{A}'$ for each modified assembly. However, this approach is very clumsy and for a complex assembly with many bars, this would be a computationally prohibitive exercise. We hereby present a computationally efficient technique which requires only a single decomposition of the equilibrium matrix to determine the necessary bars, using the states of self-stress, i.e. $\mathbf{V}_s$ in Eq. (33).

The theorem of this technique can be expressed as: "*the necessary and sufficient condition for a bar to be unconditionally necessary to maintain geometric stability in an assembly is that all the elements in the vectors of states of self-stress corresponding to that bar are equal to zero*".

(i) Proof of necessity

Since the states of self-stress is of an assembly found in the null space of $\mathbf{A}$, then by definition

$$\mathbf{AV}_{\mathrm{s}} = \mathbf{0}, \tag{43}$$

which can be expressed in terms of vectors as

$$\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k, \ldots, \mathbf{a}_b\}\{\mathbf{v}_{r+1}, \mathbf{v}_{r+2}, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_b\} = \mathbf{0}, \tag{44}$$

where $\mathbf{v}_i$ (for $i = r + 1,\ldots,b$) is a vector of self-stress. Further,

$$\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k, \ldots, \mathbf{a}_b\}\left\{\begin{matrix} v_{r+1,1} & v_{r+2,1} & \cdots & v_{i,1} & \cdots & v_{b,1} \\ v_{r+1,2} & v_{r+2,2} & \cdots & v_{i,2} & \cdots & v_{b,2} \\ \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\ v_{r+1,k} & v_{r+2,k} & \cdots & v_{i,k} & \cdots & v_{b,k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ v_{r+1,b} & v_{r+2,b} & \vdots & v_{i,b} & \vdots & v_{b,b} \end{matrix}\right\} = \mathbf{0}. \tag{45}$$

If a whole row of zeros are found in $\mathbf{V}_{\mathrm{s}}$, and the order of columns and rows in Eq. (45) is re-arranged so that this null row in $\mathbf{V}_{\mathrm{s}}$ and its corresponding column in $\mathbf{A}$ are moved to the end, then

$$\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{k-1}, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_b, \mathbf{a}_k\}\left\{\begin{matrix} v_{r+1,1} & v_{r+2,1} & \cdots & v_{i,1} & \cdots & v_{b,1} \\ v_{r+1,2} & v_{r+2,2} & \cdots & v_{i,2} & \cdots & v_{b,2} \\ \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\ v_{r+1,k-1} & v_{r+2,k-1} & \cdots & v_{i,k-1} & \cdots & v_{b,k-1} \\ v_{r+1,k+1} & v_{r+2,i+1} & \cdots & v_{i,k+1} & \cdots & v_{b,k+1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ v_{r+1,b} & v_{r+2,b} & \vdots & v_{i,b} & \cdots & v_{b,b} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}\right\} = \mathbf{0}, \tag{46}$$

which can then be partitioned and expressed more succinctly as

$$\{\mathbf{A}' \mid \mathbf{a}_k\}\left\{\frac{\mathbf{V}'}{\mathbf{0}}\right\} = \mathbf{0}, \tag{47}$$

and thus

$$\mathbf{A}'\mathbf{V}' = \mathbf{0}. \tag{48}$$

The matrix $\mathbf{A}'$ in Eq. (48) is actually the equilibrium matrix of the updated assembly after the removal of bar k, and $\mathbf{V}'$ is derived from $\mathbf{V}$ by deleting row k. It is clear that the columns of $\mathbf{V}'$ are linearly independent of each other (since they are formed from the columns of $\mathbf{V}$, which are by definition independent), and this means that there are at least $s = b - (nj - c)$ non-trivial solutions for Eq. (48), i.e. more than $b - (nj - c)$ independent self-stress states for the modified assembly. According to the definition of s, $(b - 1) - r(\mathbf{A}') > s = b - r(\mathbf{A}) = b - (nj - c)$, then $r(\mathbf{A}') \leqslant (nj - c) - 1$, and thus $m \geqslant 1$ in $\mathbf{A}'$, i.e. the new assembly is geometrically unstable. The presence of a whole row of zeros in $\mathbf{V}_s$ is thus a firm indicator of a necessary bar, the removal of which will cause the resultant assembly to be kinematically indeterminate.

(ii) Proof of sufficiency

When a necessary bar is removed from a kinematically determinate and statically indeterminate system, it becomes unstable. This means that the rank of equilibrium matrix of the reduced assembly is now less than $nj - c$, i.e. the column in $\mathbf{A}$ associated with a necessary bar must be linearly independent of all the other column. The test that vector $\mathbf{a}_k$ in equilibrium matrix $\mathbf{A}$ contributed by bar k, is linearly independent of the other columns is actually not difficult in linear algebra.

By definition, the states of self-stress $\mathbf{v}_s$ is in equilibrium with zero external load, and hence

$$\mathbf{A}\mathbf{v}_s = \mathbf{0}. \tag{49}$$

If we consider any one state of self-stress i where $i \in (1,\ldots,s)$, then Eq. (49) can be written in vectorial form as

$$v_{1,i}\mathbf{a}_1 + v_{2,i}\mathbf{a}_2 + \cdots + v_{k-1,i}\mathbf{a}_{k-1} + v_{k,i}\mathbf{a}_k + v_{k+1,i}\mathbf{a}_{k+1} + \cdots + v_{b,i}\mathbf{a}_b = 0. \tag{50}$$

If we now consider bar k as a necessary bar, then the column in $\mathbf{A}$ associated with bar k, $\mathbf{a}_k$, must be linearly independent of all the other columns. Eq. (50) can be re-arranged to move $\mathbf{a}_k$ to the right hand side, i.e.

$$v_{1,i}\mathbf{a}_1 + v_{2,i}\mathbf{a}_2 + \cdots + v_{k-1,i}\mathbf{a}_{k-1} + v_{k+1,i}\mathbf{a}_{k+1} + \cdots + v_{b,i}\mathbf{a}_b = -v_{k,i}\mathbf{a}_k, \tag{50a}$$

so that $\mathbf{a}_k$ is expressed as a linear combination of all the other columns through factors given by the coefficients $v_{j,i}$. Since $\mathbf{a}_k$ is supposed to be linearly independent, then there is no possible combinations of $[\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{k-1}, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_b]$ that can span the vector of $\mathbf{a}_k$, and hence the only possible value of $v_{k,i}$ for Eq. (50a) to hold is $v_{k,i}$=0. This conclusion can also be made for any and all of the remaining states of self-stress. The presence of a whole row of zeros in $\mathbf{V}_s$ is thus a sufficient condition to identify a bar as necessary. Further, it should be noted that the necessary bar is bar that cannot maintain self-stress.

For the example shown in Fig. 3, its two states of self-stress are calculated and listed below

$$\mathbf{v}_9 = \{-0.3755, -0.6318, 0.2563, -0.3625, -0.3625, 0.2563, 0, 0, 0.2563, 0\}^{\mathrm{T}},$$
$$\mathbf{v}_{10} = \{0.6263, 0.3663, 0.2601, -0.3679, -0.3679, 0.2601, 0, 0, 0.2601, 0\}^{\mathrm{T}}.$$

It is clear that the zero elements in the two states of self-stress corresponding to bars 7, 8 and 10 show these are necessary bars.

5. Stability of mechanism

5.1. Basic properties of mechanism

Conventionally, the definition of "mechanism" is from the viewpoint of kinematics whereby assemblies of pin-jointed bars presenting a kinematical indeterminacy are called mechanisms. Kinematic behaviour of pin-jointed bar assemblies can be obtained by analysis of the subspaces of compatibility matrix (or equilibrium matrix), i.e. $\mathbf{U}_m$ in Eq. (33).

Mechanisms belong to geometrically unstable systems, i.e. where $m = nj - c - r > 0$. As discussed in Section 4, the linear elastic stiffness matrix $\mathbf{K}_0$ of mechanism will be singular and hence, the quadratic form $\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_0\,\delta\mathbf{d}$ must be equal to zero for some displacements $\delta\mathbf{d}$.

The fact that the quadratic form takes a zero value dictates the nature of $\delta\mathbf{d}$ associated with mechanisms. Using the expression for $\mathbf{K}_0$ in Eq. (27), $\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_0\,\delta\mathbf{d}$ can be expressed as

$$\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_0\,\delta\mathbf{d} = \delta\mathbf{d}^{\mathrm{T}}\mathbf{B}_{\mathrm{L}}^{\mathrm{T}}\mathbf{M}\mathbf{B}_{\mathrm{L}}\,\delta\mathbf{d} = (\mathbf{B}_{\mathrm{L}}\,\delta\mathbf{d})^{\mathrm{T}}\mathbf{M}(\mathbf{B}_{\mathrm{L}}\,\delta\mathbf{d}). \tag{51}$$

If the assembly has no bar with zero-stiffness, then $\mathbf{M}$ is positive definitive, and the condition for the quadratic form in Eq. (51) taking zero value is that $\mathbf{B}_{\mathrm{L}}\,\delta\mathbf{d} = \mathbf{0}$. That means that the set of compatible displacements $\delta\mathbf{d}$ associated with mechanisms is a linear combination of modes of inextensional deformations (as discussed in Section 2), i.e.

$$\delta\mathbf{d} = \mathbf{u}_{r+1}\beta_{r+1} + \cdots + \mathbf{u}_{nj-c}\beta_{nj-c} = \mathbf{U}_m\boldsymbol{\beta} \tag{52}$$

where $\boldsymbol{\beta} = \{\beta_{r+1},\beta_{r+2},\ldots,\beta_{nj-c}\}^{\mathrm{T}}$ is a vector of combination coefficients.

We have seen in Section 3 that the geometric stiffness matrix may be used to judge the stability of an assembly. For a mechanism, the primary condition of stability (Case 4 in Section 3) is that

$$\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_{\mathrm{g}}\,\delta\mathbf{d} > 0 \quad \text{for } \delta\mathbf{d} = \mathbf{u}_{r+1}\beta_{r+1} + \cdots + \mathbf{u}_{nj-c}\beta_{nj-c}. \tag{53}$$

Since $\mathbf{K}_{\mathrm{g}}$ is a function of internal forces $\mathbf{t}_0$ (see Eq. (31)), then having a non-zero $\mathbf{t}_0$ is a prerequisite to satisfy $\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_{\mathrm{g}}\,\delta\mathbf{d} > 0$.

Considering Eqs. (31) and (52), Eq. (53) can be further re-written as

$$\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_{\mathrm{g}}\,\delta\mathbf{d} = \delta\mathbf{d}^{\mathrm{T}}\,\delta[\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{d})]\mathbf{t}_0 = \boldsymbol{\beta}^{\mathrm{T}}\mathbf{U}_m^{\mathrm{T}}[\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{U}_m\boldsymbol{\beta})]\mathbf{t}_0 > 0, \tag{54}$$

where $\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{U}_m\boldsymbol{\beta})$ signifies that $\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}$ is a function of $\mathbf{U}_m$.

The two ways of generating internal force to stabilise mechanisms are through self-stressing and equilibrium with external loads. In the following, we shall discuss these two kinds of stable mechanisms separately.

5.2. Prestressed mechanism

To our knowledge, the term of "*prestressed mechanism* " was first used by Pellegrino (1990) to define a class of singular configurations such as cable nets, tensegrities, etc. A determination criterion for these systems was primarily developed by Pellegrino and Calladine (1986). Consequent discussions about this criterion (Kuznetsov, 1989; Calladine and Pellegrino, 1991) stimulated improvement of the criterion to the eventual form:

$$\boldsymbol{\beta}^{\mathrm{T}}\left[\sum_{i=1}^{s}\mathbf{F}_i^{\mathrm{T}}\mathbf{U}_m\alpha_i\right]\boldsymbol{\beta} > 0, \tag{55}$$

where $\mathbf{F}_i^{\mathrm{T}}$ is the product–force matrix corresponding to the *i*th self-stress state.

Eq. (55) implies that prestress can stiffen all m inextensional mechanisms (Pellegrino, 1990). However, deduction of this criterion has so far been merely from physical explanations based on geometrically linear analysis and not through a formal theoretical development. An abstract term of "product force", which represents the out-of-balance force resulting from imposing a mechanism on an assembly with a state of self-stress, is defined and employed to check the capability equilibrium being restored (cf. Calladine and Pellegrino, 1991). We hereby offer further explanations on the stability of this kind of mechanisms, and supply the proof for Calladine and Pellegrino's criterion, but also demonstrate the extent of its validity.

For an assembly to be a prestressable mechanism, it must have $s > 0$. A particular pre-tension in an assembly can be written as a linear combination of the individual states of self-stress

$$\mathbf{t}_0 = \mathbf{v}_{r+1}\alpha_1 + \mathbf{v}_{r+2}\alpha_2 + \ldots + \mathbf{v}_b\alpha_s, \tag{56}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_s$ are combination coefficients. The expression found in Eq. (54) can be further expressed as

$$\begin{aligned}\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{U}_m\boldsymbol{\beta})\mathbf{t}_0 &= \sum_{j=1}^{m}\left[\beta_{r+j}\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{u}_{r+j})\right]\mathbf{t}_0 = \sum_{j=1}^{m}\left[\beta_{r+j}\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{u}_{r+j})\right]\sum_{i=1}^{s}\left[\alpha_i\mathbf{v}_{r+i}\right] \\ &= \sum_{j=1}^{m}\beta_{r+j}\left\{\sum_{i=1}^{s}\left[\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{u}_{r+j})\mathbf{v}_{r+i}\right]\alpha_i\right\}\end{aligned} \tag{57}$$

Defining a matrix $\mathbf{F}_i = \{\mathbf{p}_{i1}, \mathbf{p}_{i2}, \ldots, \mathbf{p}_{im}\}$ of "product forces", see Calladine and Pellegrino (1991), where $\mathbf{p}_{ij} = \mathbf{B}_{\mathrm{NL1}}{}^{\mathrm{T}}(\mathbf{u}_{r+j})\mathbf{v}_{r+i}$, Eq. (57) can be further re-written as

$$\sum_{j=1}^{m}\beta_{r+j}\left\{\sum_{i=1}^{s}\left[\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{u}_{r+j})\mathbf{v}_{r+i}\right]\alpha_i\right\} = \left[\sum_{i=1}^{s}\mathbf{F}_i\alpha_i\right]\boldsymbol{\beta}. \tag{58}$$

Noting of the symmetry of the product $\mathbf{U}_m^{\mathrm{T}}\mathbf{F}_i$ (Calladine and Pellegrino, 1991), Eq. (54) then takes the form

$$\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_{\mathrm{g}}\,\delta\mathbf{d} = \boldsymbol{\beta}^{\mathrm{T}}\mathbf{U}_m^{\mathrm{T}}\left[\sum_{i=1}^{s}\mathbf{F}_i\alpha_i\right]\boldsymbol{\beta} = \boldsymbol{\beta}^{\mathrm{T}}\left[\sum_{i=1}^{s}(\mathbf{U}_m^{\mathrm{T}}\mathbf{F}_i)\alpha_i\right]\boldsymbol{\beta} = \boldsymbol{\beta}^{\mathrm{T}}\left[\sum_{i=1}^{s}\mathbf{F}_i^{\mathrm{T}}\mathbf{U}_m\alpha_i\right]\boldsymbol{\beta} > 0, \tag{59}$$

which thus proves the criterion (Eq. (55)) given by Calladine and Pellegrino. Although the expression $\boldsymbol{\beta}^{\mathrm{T}}\left[\sum_{i=1}^{s}\mathbf{F}_i^{\mathrm{T}}\mathbf{U}_m\alpha_i\right]\boldsymbol{\beta} > 0$ is an equivalent expression of $\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_{\mathrm{g}}\,\delta\mathbf{d} > 0$, the former expression is the form that presents a clear description of the effect of feasible self-stress parameters α_i in stabilising a mechanism.

From the analysis above, any set of feasible solution $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_s\}^{\mathrm{T}}$ which satisfies $\boldsymbol{\beta}^{\mathrm{T}}\left[\sum_{i=1}^{s}\mathbf{F}_i^{\mathrm{T}}U_m\alpha_i\right]\boldsymbol{\beta} > 0$ can stiffen all possible inextensional mechanisms. However, following the stability condition of Case 4 in Section 3, arbitrarily increasing α_i can cause $\delta\mathbf{d}^{\mathrm{T}}(\mathbf{K}_0 + \mathbf{K}_{\mathrm{g}})\,\delta\mathbf{d} < 0$ for extensional displacements (i.e. not mechanisms). This unstable phenomenon can be illustrated by the simple prestressed mechanism shown in Fig. 4, where a sufficiently high level of prestress in the assembly causes buckling in the compression bar, and thereby resulting in loss of prestress altogether and therefore instigating instability. This type of instability, however, is akin to stability condition of Case 3 in Section 3, which is not the primary subject in this paper.

5.3. Loaded mechanism

Self-stress to stiffen mechanisms is only available when there exists non-trivial solutions to the equilibrium equation $\mathbf{A}\mathbf{t}_0 = \mathbf{0}$. As mentioned above, external load is another way to stiffen and stabilise a mechanism even if the structural geometry is not amenable to prestress, such as the example in Fig. 1(e).

Just as prestress is a linear combination of the independent states of self-stress (Eq. (56)), a set of bar forces $\mathbf{t}_l$ in equilibrium external loads must yield

$$\mathbf{t}_l = \mathbf{v}_1\alpha_1 + \mathbf{v}_2\alpha_2 + \cdots + \mathbf{v}_r\alpha_r \tag{60}$$

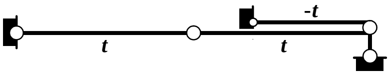

Fig. 4. A prestressed mechanism in which the uniform state of self-stress involves tension in two bars and compression in the third bar. This assembly has conditional stability even though Eq. (59) is satisfied (the three bars are supposed to be collinear).

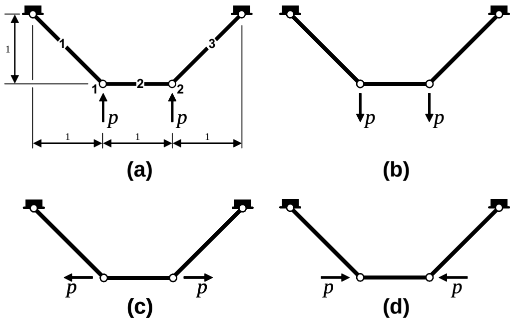

Fig. 5. Four equilibrium states for a finite mechanism under different external loads.

where $\alpha_l, \alpha_2, \ldots, \alpha_r$ are combination coefficients. Similarly, not every $\mathbf{t}_l$ is capable of stabilising a mechanism. Fig. 5 shows a finite mechanism under four different equilibrium states. With the external load exciting the finite mechanism, Case 1 is unstable. Reversing the load directions for Case 2 induces the reverse internal force, which actually stiffens the finite mechanism so that the assembly is now in stable equilibrium. Cases 3 and 4, however, are not easily judged as to whether they are stable or not. Consequently, a criterion for determining stability of loaded mechanism will now be developed.

Substituting $\mathbf{t}_0$ of Eq. (54) with $\mathbf{t}_l$ in Eq. (60),

$$\begin{aligned}\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{U}_m\boldsymbol{\beta})\mathbf{t}_l &= \sum_{j=1}^{m}\left[\beta_{r+j}\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{u}_{r+j})\right]\mathbf{t}_l = \sum_{j=1}^{m}\left[\beta_{r+j}\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{u}_{r+j})\right]\sum_{i=1}^{r}[\alpha_i\mathbf{v}_i] \\ &= \sum_{j=1}^{m}\beta_{r+j}\left\{\sum_{i=1}^{r}\left[\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{u}_{r+j})\mathbf{v}_i\right]\alpha_i\right\}.\end{aligned} \tag{61}$$

Again, defining a matrix $\mathbf{F}_i = \{\mathbf{p}_{i1}, \mathbf{p}_{i2}, \ldots, \mathbf{p}_{im}\}$ of "product forces" similar to that of prestressed mechanism, and $\mathbf{p}_{ij} = \mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{u}_{r+j})\mathbf{v}_i$, Eq. (61) can be further written as

$$\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{U}_m\boldsymbol{\beta})\mathbf{t}_l = \left[\sum_{i=1}^{r}\mathbf{F}_i\alpha_i\right]\boldsymbol{\beta}. \tag{62}$$

Substitution of Eq. (62) into Eq. (54) and again noting the symmetry of the product forces $\mathbf{U}_m^{\mathrm{T}}\mathbf{F}_i$, the stability condition takes the form

$$\delta\mathbf{d}^{\mathrm{T}}\mathbf{K}_{\mathrm{g}}\,\delta\mathbf{d} = \boldsymbol{\beta}^{\mathrm{T}}\mathbf{U}_m^{\mathrm{T}}\left[\sum_{i=1}^{r}\mathbf{F}_i\alpha_i\right]\boldsymbol{\beta} = \boldsymbol{\beta}^{\mathrm{T}}\left[\sum_{i=1}^{r}(\mathbf{U}_m^{\mathrm{T}}\mathbf{F}_i)\alpha_i\right]\boldsymbol{\beta} = \boldsymbol{\beta}^{\mathrm{T}}\left[\sum_{i=1}^{r}\mathbf{F}_i^{\mathrm{T}}\mathbf{U}_m\alpha_i\right]\boldsymbol{\beta} > 0. \tag{63}$$

Although the final form of Eq. (63) appears to be just as Eq. (55) for prestressed mechanisms, it should be noted that the "product forces", $\mathbf{F}_i$, in the two equations have different meanings.

We can carry out stability analysis of illustrative examples shown in Fig. 5 and prescribe $p = 1$. Through SVD on the equilibrium matrix of assembly, the static and kinematic parameters of this assembly can be obtained as follows:

$$r = 3; \quad m = 1; \quad s = 0; \quad \mathbf{U}_m = \mathbf{u}_4 = \{-0.5, -0.5, -0.5, 0.5\}^{\mathrm{T}};$$

Table 2
Stability determination for the four equilibratory systems shown in Fig. 5

Case	$\mathbf{t}_0$	α_1	α_2	α_3	$\beta^{\mathrm{T}}\left[\sum_{i=1}^{s}\mathbf{F}_i^{\mathrm{T}}\mathbf{U}_m\alpha_i\right]\boldsymbol{\beta}$	Result
(a)	$\{-\sqrt{2},-1,-\sqrt{2}\}^{\mathrm{T}}$	0.2008	0	−2.2270	$-1.5\beta_4^2 < 0$	Unstable
(b)	$\{\sqrt{2},1,\sqrt{2}\}^{\mathrm{T}}$	−0.2008	0	2.2270	$1.5\beta_4^2 > 0$	Stable
(c)	$\{0,1,0\}^{\mathrm{T}}$	0.8507	0	0.5257	$0.5\beta_4^2 > 0$	Stable
(d)	$\{0,-1,0\}^{\mathrm{T}}$	−0.8507	0	−0.5257	$-0.5\beta_4^2 < 0$	Unstable

α_i are obtained by $\mathbf{V}_r^{-1}\mathbf{t}_0$.

$$\mathbf{V} = \mathbf{V}_r = \{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\} = \begin{Bmatrix} -0.3717 & 0.7071 & 0.6015 \\ 0.8507 & 0 & 0.5257 \\ -0.3717 & -0.7071 & 0.6015 \end{Bmatrix}.$$

Substituting $\mathbf{u}_4$ into Eq. (22), gives

$$\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{u}_4) = \begin{Bmatrix} -\frac{0.5}{\sqrt{2}} & 0 & 0 \\ -\frac{0.5}{\sqrt{2}} & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{0.5}{\sqrt{2}} \\ 0 & \frac{1}{2} & \frac{0.5}{\sqrt{2}} \end{Bmatrix}.$$

Since $m = 1$, matrix $\mathbf{F}_i^{\mathrm{T}}\mathbf{U}_m$ in Eq. (63) degenerates to a scalar, and can be obtained by

$$\mathbf{F}_1^{\mathrm{T}}\mathbf{U}_m = (\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{u}_4)\mathbf{v}_1)^{\mathrm{T}}\mathbf{U}_m = 0.1625;$$
$$\mathbf{F}_2^{\mathrm{T}}\mathbf{U}_m = (\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{u}_4)\mathbf{v}_2)^{\mathrm{T}}\mathbf{U}_m = 0;$$
$$\mathbf{F}_3^{\mathrm{T}}\mathbf{U}_m = (\mathbf{B}_{\mathrm{NL1}}^{\mathrm{T}}(\mathbf{u}_4)\mathbf{v}_3)^{\mathrm{T}}\mathbf{U}_m = 0.6882.$$

The stability of the four equilibrium systems in Fig. 5 can thus be determined, and the results are as listed in Table 2.

6. Stability of infinitesimal mechanisms

In kinematic terms, an infinitesimal mechanism is defined as a system that possesses "virtual mobility" but no actual kinematic mobility, i.e. with unique configuration (Kuznetsov, 1999). The simplest example of an infinitesimal mechanism is the von Mises trusses with collinear pins as shown in Fig. 6. While such assemblies are classified as "mechanisms" since they have non-zero number of inextensional mechanisms (i.e. $m > 0$ just like that of geometrically unstable systems), it is paradoxical that these von Mises trusses in Fig. 6 are obviously stable according to Simitses' concept (see first paragraph of Section 1) even without any existence of prestress.

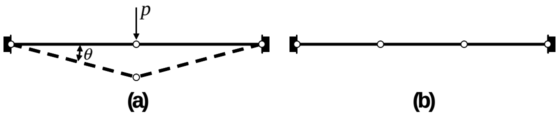

Fig. 6. von Mises trusses with collinear joints.

Singular behaviour of infinitesimal mechanisms was attended as early as the works of Maxwell (1890). Calladine raised the matter again in 1978, talking about the mechanism behaviour of tensegrity structures. A concept of "order of infinitesimal mechanisms" was also subsequently put forward. In the beginning, only first-order infinitesimal mechanisms with a single degree of indeterminacy were investigated (Tarnai, 1980; Pellegrino and Calladine, 1986). Later, more complex systems with higher-order infinitesimal mobility and higher degree of indeterminacy were addressed (Calladine and Pellegrino, 1991; Kuznetsov, 1991). Koiter (Tarnai, 1989) suggested a general method for evaluating the order of infinitesimal mobility based on the nonlinear theory of elastic stability, and a formal definition of the order of infinitesimal mechanisms was formulated by Tarnai (1989). However, understanding of this kind of assembly is not yet fully consistent.

Since infinitesimal mechanisms have a singular linear elastic stiffness matrix, $\mathbf{K}_0$ then regardless of the possible contribution from the geometrical stiffness matrix, $\mathbf{K}_g$, through internal force effects, only Case 5 of stability conditions (see Section 3) is suitable for judging their stability. This means that higher than second-order variations of potential energy should be probed.

We illustrate this with the two-bar system shown in Fig. 6(a). A downward force P at the middle joint causes a vertical nodal displacement (i.e. in the direction of the inextensional mechanism) and the corresponding rotation θ of bars. If we take the rotation θ as the generalised displacement, the total potential energy of system can be written as

$$\Pi = EAL(1/\cos\theta - 1)^2 - PL\tan\theta, \tag{64}$$

where E, A, and L are respectively the Young's modulus, cross-sectional area and unstrained length of two bars.

By differentiating both sides of Eq. (64), the equilibrium relationship of system can be obtained by

$$\frac{\partial\Pi}{\partial\theta} = 2EAL(1/\cos\theta - 1)\sin\theta/\cos^2\theta - PL/\cos^2\theta = 0, \tag{65}$$

and hence

$$P = 2EA(1/\cos\theta - 1)\sin\theta, \tag{66}$$

Higher-order variations of Π at initial equilibrium state with $\theta = 0$ and $P = 0$ are found as:

$$\delta^2\Pi\big|_{\theta=0} = \left(\frac{1}{2!}\frac{\partial^2\Pi}{\partial\theta^2}\bigg|_{\theta=0}\right)\delta\theta^2 = \left[EAL(1-\cos^3\theta)/\cos^4\theta\big|_{\theta=0}\right]\delta\theta^2 = 0, \tag{67}$$

$$\delta^3\Pi\big|_{\theta=0} = \left(\frac{1}{3!}\frac{\partial^3\Pi}{\partial\theta^3}\bigg|_{\theta=0}\right)\delta\theta^3 = \left[\left(EAL\sin\theta(2-\cos^3\theta)/\cos^5\theta\right)\big|_{\theta=0}\right]\delta\theta^3 = 0, \tag{68}$$

$$\begin{aligned}\delta^4\Pi\big|_{\theta=0} &= \left(\frac{1}{4!}\frac{\partial^4\Pi}{\partial\theta^4}\bigg|_{\theta=0}\right)\delta\theta^4 = \left[\left(\frac{1}{12}EAL\left(-28\cos^2\theta - 12\cos^3\theta + 7\cos^5\theta + 36\right)/\cos^6\theta\right)\bigg|_{\theta=0}\right]\delta\theta^4 \\ &= \frac{1}{4}EAL\,\delta\theta^4 > 0.\end{aligned} \tag{69}$$

Hence, according to the energy criterion of structural stability given in Section 2, although $\delta^2\Pi = 0$, this infinitesimal mechanism is still stable because $\delta^3\Pi = 0$ and $\delta^4\Pi > 0$.

Analysis of higher-order variation of potential energy should thus be carried out to reveal the singular nature of infinitesimal mechanisms, and knowledge of second-order variation $\delta^2\Pi$ alone is obviously insufficient. Salerno (1992) has numerically investigated more complex infinitesimal mechanisms from the viewpoint of structural stability. However, analysis of higher-order variation of potential energy is more complicated than that of $\delta^2\Pi$. While $\delta^2\Pi$ is only a quadratic form, and its properties can be obtained through analysis of the stiffness matrix by means of conventional theory of linear algebra, higher-order variations of potential energy demand much more difficult analyses.

It should be noted that Kuznetsov (1999, 2000) presented a novel perspective about the physical realisability and exact computability of such singular system based on mathematic concepts of structural stability (unrelated to structures). In his viewpoints, as the degenerated configurations of geometrically invariant or variant systems, infinitesimal mechanisms are always sensitive to the minute changes of control parameters such as linear and angular sizes of the structural members (i.e. lengths of bars for pin-jointed bar assemblies). It means that any infinitesimal changes of lengths of bars, e.g. those in Fig. 6, will essentially lead those assemblies to be geometrically invariant (Fig. 6a) or variant (Fig. 6b). Considering the exact values of control parameters can never be known in both real situation and numerical calculation, unprestressed or unprestressable first-order mechanisms and all higher-order infinitesimal mechanisms are concluded to be physically unrealisable and noncomputable except for symbolic or integer calculations. The only physically realizable configuration is first-order infinitesimal mechanisms possessing prestress of finite magnitude as those discussed in Section 5, in consideration that physical prestress can override all geometric imperfections, including lack of precision in the member sizes and in the process of assembly.

Although research on infinitesimal mechanisms is still a formidable problem in spite of their realisability and computability, use of the theory of structural stability is undeniably the best way to recognise and classify this kind of singular assemblies. At the very least, we are able to theoretically see that the stability condition for prestressed infinitesimal mechanism is obviously different to that without internal forces. The former lies in discussions of second-order variation of potential energy, but the latter depends on the higher-order.

7. Discussion

Geometry and topology of assembly, stiffness of members and internal forces are three aspects all affecting the stability of an equilibratory system. Conventional analysis of structural stability is mostly focused on effects of internal forces, but does not regard the first two factors. On the other hand, "static-kinematic" analysis does not usually treat the issue of stability. All three factors have been considered at the same time in this paper.

From the viewpoint of energy criterion, intrinsic stability of assembly is equivalent to having a positive definitive linear elastic stiffness matrix for the assembly. It should be noted that Maxwell's rule and the criterion based on equilibrium matrix are insufficient, since they ignore effects of member stiffness. However, if there is not a bar with zero stiffness in assembly, the rank of the equilibrium matrix can perfectly reflect the properties of linear elastic stiffness matrix, whether it is positive definite or not. As a supplementary, this paper takes a close look at the effects of member stiffness, which is usually neglected, to the intrinsic stability of system, and mathematically shows the equivalence between removal of bar and zero stiffness of the same bar. Furthermore, investigations of "necessary bar" extends for the first time the analysis of "intrinsic stability" from "assembly" level to "member" level.

Stability of mechanisms cannot be comprehended properly without treating the three aspects affecting stability of assembly and considering their interactions. Effective stabilisation of mechanisms comes through reinforcement from internal forces, which are not only caused by self-stress but can also arise from loading. The stability criterion of prestressed and loaded mechanisms can be proven and understood from energy criterion and the analytic expression of the tangential stiffness matrix. If stability from prestress is well illustrated by those prestressable structures such as cable nets or tensegrity systems, then the simple suspended cable roof under heavy dead loads can be regarded as an equally good example of a loaded mechanism. In fact, such suspended systems rely on the stability due to heavy dead load to sustain changeable live loads effectively.

In many cases, properties of the second-order variation of potential energy are adequate to show the behavior of stability. However, for stability of infinitesimal mechanisms devoid of internal forces, higher-order variations of potential energy must be investigated. Hence, it is clear that stabilities of infinitesimal mechanisms with, and without, internal forces are fundamentally different in mathematical terms. Furthermore, studies on stability of infinitesimal mechanisms cannot be conclusive if the focus is only on second-order variation of potential energy.

Undeniably, nonlinear theory of structural stability has explained many non-generic stability phenomena in this paper, which will be helpful to establish a unified classification of stability of pin-jointed bar assemblies.

Acknowledgments

The work of the first author was supported by the National Natural Science Foundation of China (grant no. 50108014) and the China Scholarship Council. The authors also express their thanks to Dr. Q.Z. Xiao (Cardiff School of Engineering) for his advice.

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